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LETTER TO THE EDITOR

Closed-form relations for phase-space representatives of spin-J operators

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Abstract. By introducing the holomorphic representation we obtain, in closed form, relations between the c-number functions, which represent an operator in spin-J space, without recurring to expansions in spherical harmonics. In particular, we obtain the diagonal representative from the diagonal matrix element through an integral relation. New analogies between these and the spin (Bloch) coherent states are established which avoid the introduction of Schwinger bosons, and the consequent projection to the physical subspace, by introducing a generalized Fourier transformation in spin-J space in a way similar to the treatment by Cahill and Glauber for boson coherent states. Also, we study the generalization to the spin case of the s-ordering of Cahill and Glauber.

The spin coherent states [1–3] $|\alpha\rangle = \hat{R}(\alpha)|JJ\rangle$ can be defined in terms of the unitary operator

$$\hat{R}(\alpha) = e^{\alpha J_-} e^{[\ln 1/(1+|\alpha|^2)]J_z} e^{-\alpha^* J_+}$$

where $\alpha = \tan\frac{\theta}{2}e^{i\phi}$, and θ and ϕ are the spherical angles, yielding

$$|\alpha\rangle = \frac{1}{(1+|\alpha|^2)^J} e^{\alpha \hat{J}_-}|JJ\rangle. \tag{1}$$

Two types of c-number representatives for a spin operator $\hat{F}$ are usually defined. One is given by the diagonal matrix element [2, 1] $F(\alpha^*, \alpha) = \langle\alpha|\hat{F}|\alpha\rangle$. The other is the diagonal representative $f(\alpha, \alpha^*)$, defined by [1–3]

$$\hat{F} = \frac{2J+1}{\pi}\int \frac{d^2\alpha}{(1+|\alpha|^2)^2}|\alpha\rangle f(\alpha, \alpha^*)\langle\alpha|. \tag{2}$$

These representatives are very useful in calculations and their properties and the relations between them are important.

We note first that the coherent state $|\alpha\rangle$, given by equation (1), depends not only on the variable α but also on the variable α^*, which should be considered independent. As for bosons, it is, therefore, convenient to introduce the holomorphic representation [4], defining the new states $\|\alpha\rangle = e^{\alpha\hat{J}_-}|JJ\rangle$ which depend only on α and not α^*. Their overlap is $\langle\beta\|\hat{1}\|\alpha\rangle = (1+\beta^*\alpha)^{2J}$. In connection with these states, it is convenient to use the diagonal representative $\bar{f}(\alpha, \alpha^*)$ defined by

$$\hat{F} = \frac{2J+1}{\pi}\int d^2\alpha\, \|\alpha\rangle \bar{f}(\alpha, \alpha^*)\langle\alpha\| \tag{3}$$

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and related to $f(\alpha,\alpha^*)$ by $f(\alpha,\alpha^*)=(1+|\alpha|^2)^{2(J+1)}\bar{f}(\alpha,\alpha^*)$, since $\|\alpha\rangle=(1+|\alpha|^2)^J|\alpha\rangle$.

Products of operators appear in most calculations and it is important to know their representatives. The normalized matrix element of the product of two operators $\hat{F}_{21}=\hat{F}_2\hat{F}_1$ has been obtained previously and, inserting a resolution of the identity, can be written as

$$F_{21}(\alpha^*,\alpha)=\frac{2J+1}{\pi}\int\frac{\mathrm{d}^2\gamma}{(1+|\gamma|^2)^2}F_2(\alpha^*,\gamma)\left(\frac{(1+\alpha^*\gamma)(1+\gamma^*\alpha)}{(1+|\alpha|^2)(1+|\gamma|^2)}\right)^{2J}F_1(\gamma^*,\alpha) \tag{4}$$

where $F(\gamma^*,\alpha)=\langle\gamma\|\hat{F}\|\alpha\rangle/\langle\gamma\|\hat{1}\|\alpha\rangle$. The diagonal representative is obtained similarly, removing the end bra and ket states, as

$$f_{21}(\alpha,\alpha^*)=\frac{2J+1}{\pi}\int\frac{\mathrm{d}^2\gamma}{(1+|\gamma|^2)^2}f_2(\alpha,\gamma^*)\left(\frac{(1+|\alpha|^2)(1+|\gamma|^2)}{(1+\gamma^*\alpha)(1+\alpha^*\gamma)}\right)^{2(J+1)}f_1(\gamma,\alpha^*). \tag{5}$$

Taking $\hat{F}_2=\hat{1}$ in equation (4), we are led to the one-variable delta-function [1]

$$\delta(\tau^*,\alpha^*)=\left(\frac{1+\alpha^*\tau}{1+\tau^*\tau}\right)^{2J}. \tag{6}$$

However, the same choice in equation (5) gives us a new one-variable delta-function

$$\delta'(\gamma,\alpha)=\left(\frac{1+|\gamma|^2}{1+\gamma^*\alpha}\right)^{2(J+1)} \tag{7}$$

where $\delta'(\gamma,\alpha)$ is defined similarly to $\delta(\gamma,\alpha)$ (cf equation (3.18) of [1])

$$c(\alpha)=\frac{2J+1}{\pi}\int\frac{\mathrm{d}^2\gamma}{(1+|\gamma|^2)^2}\delta'(\gamma,\alpha)c(\gamma). \tag{8}$$

The delta-function $\delta(\gamma,\alpha)$ also satisfies equation (8) for $c(\gamma)$ of the general form $P(\gamma)/(1+|\gamma|^2)^J$, where $P(\gamma)$ is an arbitrary polynomial in γ with terms up to γ^{2J} [1] (which is the form of the overlap between an arbitrary state and a spin coherent state). On the other hand, the delta-function $\delta'(\gamma,\alpha)$ satisfies equation (8) for $c(\gamma)$, a diagonal representative $\bar{c}(\gamma,\alpha^*)$, as defined by equation (3). This new one-variable delta-function is very useful in calculations [5].

The two representatives $F(\alpha^*,\alpha)$ and $f(\alpha,\alpha^*)$ are related through an equation of the type [6, 7]

$$F(\alpha^*,\alpha)=\int\frac{4\,\mathrm{d}^2\gamma}{(1+|\gamma|^2)^2}k_+(\gamma,\gamma^*;\alpha,\alpha^*)f(\gamma,\gamma^*) \tag{9}$$

with kernel [6, 3]

$$k_+=\frac{2J+1}{4\pi}|\langle\alpha|\gamma\rangle|^2.$$

Using the expansions in spherical harmonics for both $F(\alpha^*,\alpha)$ and $f(\alpha,\alpha^*)$, it has been possible to invert equation (9) and to express $f(\alpha,\alpha^*)$ in terms of $F(\alpha^*,\alpha)$, finding the inverse relation [3, 6, 7]

$$f(\alpha,\alpha^*)=\int\frac{4\,\mathrm{d}^2\gamma}{(1+|\gamma|^2)^2}k_-(\gamma,\gamma^*;\alpha,\alpha^*)F(\gamma^*,\gamma). \tag{10}$$

The kernel k_- has been given in terms of an expansion in spherical harmonics [6]. We will show next that it is possible to obtain this kernel in closed form, avoiding the expansions in spherical harmonics.

The relation expressing the diagonal representative in terms of the matrix element can be most simply obtained by writing $\hat{F} = \hat{1}\hat{F}\hat{1}$, using equation (3) for $\hat{F}$ on the left-hand side and a similar resolution of unity for $\hat{1}$ on the right-hand side and treating the complex variables α and α^* as independent. One then arrives at

$$\bar{f}(\alpha, \alpha^*) = \frac{2J+1}{\pi} \int \mathrm{d}^2\beta \, \frac{\langle\beta\|\hat{F}\|\beta\rangle}{[(1+\alpha^*\beta)(1+\beta^*\alpha)]^{2(J+1)}}. \tag{11}$$

In this way we are able to find a closed-form expression for the relation between the diagonal representative and the matrix element, without resorting to an expansion in spherical harmonics. Note that in the limit $J \to \infty$ [3], this equation reproduces the boson result [8]

$$f_{\mathrm{boson}}(\alpha, \alpha^*) = \mathrm{e}^{|\alpha|^2} \int \frac{\mathrm{d}^2\beta}{\pi} \mathrm{e}^{|\beta|^2} \mathrm{e}^{-\alpha^*\beta+\beta^*\alpha} \langle -\beta|\hat{F}|\beta\rangle. \tag{12}$$

The expansion in spherical harmonics for the kernel k_- can be obtained using the δ'-function equation (7) [5].

We will now generalize, to the spin coherent states, the formalism developed by Cahill and Glauber for the boson coherent states. Our starting point is the following theorem:

$$\frac{2J+1}{16\pi^2} \int \mathrm{d}g \, \langle\alpha|\hat{D}(g^{-1})|\beta\rangle\langle\gamma|\hat{D}(g)|\delta\rangle = \langle\alpha|\delta\rangle\langle\gamma|\beta\rangle \tag{13}$$

where $\hat{D}(g)$ is an irreducible representation of the rotation group for spin J, $\hat{D}(g^{-1}) = \hat{D}^{-1}(g)$ and $\mathrm{d}g$ is the invariant measure for the rotation group [9]. This theorem can be obtained either by direct evaluation or using Schur's lemma for an irreducible representation of the rotation group using the states $|Jm\rangle$ as a discrete basis [9]. From this theorem, it follows that

$$\hat{F} = \frac{2J+1}{16\pi^2} \int \mathrm{d}g \, \mathrm{Tr}(\hat{F}\hat{D}(g^{-1}))\hat{D}(g). \tag{14}$$

Defining a new representative of an operator $\hat{F}$ as $\tilde{f}(g) = \mathrm{Tr}(\hat{F}\hat{D}(g^{-1}))$, we verify that the operator $\hat{F}$ can be given the following Fourier decomposition:

$$\hat{F} = \frac{2J+1}{16\pi^2} \int \mathrm{d}g \, \tilde{f}(g)\hat{D}(g). \tag{15}$$

This treatment is similar to the one performed for the boson coherent states [10]. It is easy to show that [5] $\langle\alpha\|\hat{D}(g)\|\alpha\rangle = [\mathcal{D}(g;\alpha^*,\alpha)]^{2J}$ and the diagonal representative $\bar{d}(g;\alpha,\alpha^*) = [\mathcal{D}(g^{-1};\alpha^*,\alpha)]^{-2(J+1)}$, where $\mathcal{D}(g;\alpha^*,\alpha)$ is the matrix element for the spin-$\frac{1}{2}$ representation. The matrix element is given by

$$\langle\alpha|\hat{F}|\alpha\rangle = \frac{2J+1}{16\pi^2} \int \mathrm{d}g \, \tilde{f}(g)\langle\alpha|\hat{D}(g)|\alpha\rangle. \tag{16}$$

Using the diagonal representative of $\hat{D}(g)$, it follows, from its definition, that $\tilde{f}(g)$ is given by

$$\tilde{f}(g) = \frac{2J+1}{\pi} \int \frac{\mathrm{d}^2\alpha}{(1+|\alpha|^2)^2} \bar{d}(g^{-1};\alpha,\alpha^*)\langle\alpha|\hat{F}|\alpha\rangle. \tag{17}$$

Equations (16) and (17) constitute a pair of Fourier-transformation formulae.

Similarly to equation (2), we can also expand $\hat{F}$ as

$$\hat{F} = \frac{2J+1}{\pi} \int \frac{\mathrm{d}^2\alpha}{(1+|\alpha|^2)^2} \hat{H}(\alpha, \alpha^*) \langle\alpha|\hat{F}|\alpha\rangle \tag{18}$$

which defines the operator $\hat{H}(\alpha, \alpha^*)$. One must have $\langle\beta|\hat{H}(\alpha, \alpha^*)|\beta\rangle = \delta(\alpha - \beta, \alpha^* - \beta^*)$, the two-variable delta-function.

Another pair of transformation formulae follows if one assumes that the relation between representatives of different operators is preserved by the Fourier transformation. One must then have

$$f(\alpha, \alpha^*) = \frac{2J+1}{16\pi^2} \int \mathrm{d}g\, \tilde{f}(g)\, d(g; \alpha, \alpha^*) \tag{19}$$

from which one immediately retrieves equation (15). Using the diagonal representative for the operator $\hat{F}$, one finds that

$$\tilde{f}(g) = \frac{2J+1}{\pi} \int \frac{\mathrm{d}^2\alpha}{(1+|\alpha|^2)^2} \langle\alpha|\hat{D}(g^{-1}|\alpha\rangle f(\alpha, \alpha^*). \tag{20}$$

Inserting equation (17) into (16) or (20) into (19), one concludes that the two-variable delta-function is given by

$$\delta(\beta - \alpha, \beta^* - \alpha^*) = \frac{2J+1}{16\pi^2} \int \mathrm{d}g\, \langle\alpha|\hat{D}(g)|\alpha\rangle\, d(g^{-1}; \beta, \beta^*). \tag{21}$$

This delta-function is also the diagonal representative (at β, β^*) of the projector $|\alpha\rangle\langle\alpha|$. Similarly, introducing equation (16) into (17) or (19) into (20), one concludes that the group-space delta-function is given by $\delta(g', g) = \mathrm{Tr}(\hat{D}(g^{-1})\hat{D}(g'))$.

In general, the trace of the product of two operators is given by

$$\mathrm{Tr}(\hat{F}_1\hat{F}_2) = \frac{2J+1}{4\pi} \int \mathrm{d}\Omega(\alpha)\, f_1(\alpha, \alpha^*) \langle\alpha|\hat{F}_2|\alpha\rangle \tag{22}$$

$$= \frac{2J+1}{16\pi^2} \int \mathrm{d}g\, \tilde{f}_1(g) \tilde{f}_2(g^{-1}). \tag{23}$$

From equation (22), together with (18), it is easy to see that $f(\alpha, \alpha^*) = \mathrm{Tr}(\hat{F}\hat{H}(\alpha, \alpha^*))$. Using equation (14), we obtain that

$$\hat{H}(\alpha, \alpha^*) = \frac{2J+1}{16\pi^2} \int \mathrm{d}g\, d(g^{-1}; \alpha, \alpha^*)\, \hat{D}(g). \tag{24}$$

It is easy to find that the diagonal representative $h(\alpha, \alpha^*; \beta, \beta^*) = k_-(\alpha, \alpha^*; \beta, \beta^*)$ [5].

The operator $\hat{D}(g)$ is a general spin-J irreducible representation of the rotation group. Two representations are particularly important. The first can be written in the form $\hat{t}(\alpha, \phi) = \hat{R}(\alpha)\mathrm{e}^{\mathrm{i}\phi\hat{J}_z}\hat{R}^{-1}(\alpha)$ showing that this representation is associated with the conjugacy classes of the rotation group (or of its covering group). The second representation is associated with (right) cosets of the subgroup $U(1)$ generated by $\hat{J}_z$ in $SU(2)$ and is defined by $\hat{D}(\xi, \chi) = \hat{R}(\xi)\mathrm{e}^{\mathrm{i}\chi\hat{J}_z}$.

We define the operators $\hat{t}_N(\alpha,\phi) = \hat{t}(\alpha,\phi)/Z(\phi)$ which have unit trace by construction and where $Z(\phi)$ is the trace of $\hat{t}(\alpha,\phi)$. Their eigenvectors are $\hat{R}(\alpha)|Jm\rangle$ with eigenvalue $e^{i\phi m}/Z(\phi)$. One finds that

$$\lim_{u\to-\infty} \hat{t}_N(\alpha, -(\phi - iu)) = |\alpha\rangle\langle\alpha| \tag{25}$$

and therefore, in this limit, this operator yields the matrix element by $\langle\alpha|\hat{F}|\alpha\rangle = \mathrm{Tr}(\hat{F}|\alpha\rangle\langle\alpha|)$. We also find that

$$\lim_{u\to-\infty} e^{-i(2J+1)(\phi-iu)} t_N(\alpha, \phi - iu; \tau, \tau^*) = h(\alpha, \alpha^*; \tau, \tau^*) \tag{26}$$

giving the diagonal representative of the operator $\hat{H}(\alpha,\alpha^*)$. Using these two representations, we find, by these limiting procedures, the projector $|\alpha\rangle\langle\alpha|$ which yields the diagonal matrix element of an arbitrary operator and, in the other limiting procedure, the diagonal representative of the operator $\hat{H}$ which gives the diagonal representative of an arbitrary operator.

Using the Holstein–Primakoff [11] transformation in the limit $J \to \infty$, one has $\hat{J}_-/\sqrt{2J} \to a^\dagger$ and $\hat{J}_+/\sqrt{2J} \to a$. Quantities involving $\hat{J}_z$ can be treated in two different ways: using either $\hat{J}_z/J \to 1$ or $\hat{J}_z - J = -\hat{N}$. Two Lie group structures can then be considered: one having a, $a^\dagger$ and the identity as generators, and another having a, $a^\dagger$, $\hat{N} = a^\dagger a$ and the identity as generators. With the $\hat{J}_z/J \to \hat{1}$ procedure, the variable χ in the coset representation becomes trivial and we find the representation based on the operators $\hat{D}(\alpha)$ as in [10]. With the $\hat{J}_z - J = -\hat{N}$ procedure, the conjugacy class representation becomes the representation based on the operator $\hat{T}(\alpha, s)$ of [10]. Defining $e^u = (1+s)/(1-s)$, the operator $\hat{T}(0,s)$ becomes $(1+e^u)(-e^u)^{\hat{N}}$. For $u<0$, one has

$$\lim_{J\to\infty} \frac{e^{-i(\phi-iu)\hat{J}_z}}{Z(\phi - iu)} = (1 - e^{u+i\phi})e^{(u+i\phi)\hat{N}} \tag{27}$$

which, for $\phi = \pm\pi$, is $\hat{T}(0,s)$ and

$$\lim_{J\to\infty} -e^{-i(2J+1)(\phi-iu)} \frac{e^{i(\phi-iu)\hat{J}_z}}{Z(\phi - iu)} = (1 - e^{-(u+i\phi)})e^{-(u+i\phi)\hat{N}} \tag{28}$$

which, for $\phi = \pm\pi$, is $\hat{T}(0,-s)$. In this manner, we retrieve the results valid for bosons, as obtained by Cahill and Glauber.

This analysis allows us to define the equivalent of s-ordering for the rotation group or, in general, a Lie group. The representative of an operator is defined by the trace of that operator with the element $\hat{D}(g) = \hat{R}(\alpha)e^{i\phi\hat{J}_z}\hat{R}^{-1}(\alpha)$ in the conjugacy class representation. The ordering parameter $s = \tanh\frac{u}{2}$ is defined by the imaginary part of the variable ϕ, associated with the operator $\hat{J}_z$, the generator of the subgroup $U(1)$ of $SU(2)$. The extension of ϕ to complex values takes us outside the group. In the limit $u \to \pm\infty$, these operators become singular and we find a projector or the operator associated with the diagonal representative. The Fourier transform of the representative of an operator is given by the trace of the operator with the element $\hat{D}(g) = \hat{R}(\xi)e^{i\chi\hat{J}_z}$ in the coset representation. The kernel of this Fourier transformation is given by the trace of the product of two group elements; one written in the conjugacy class representation and the other in the coset representation. For *finite* J, *this amounts to a simple change of variables.* The complex extension of $\hat{R}(\xi)e^{i\chi\hat{J}_z}$ is obtained from $\hat{R}(\alpha)e^{i(\phi-iu)\hat{J}_z}\hat{R}^{-1}(\alpha)$ by Fourier transformation using this kernel.

These representatives depend, in general, on α and ϕ and their Fourier transforms, in general, on ξ and χ and not only on α or ξ. The same applies to the integral giving the trace of the product of two operators. For finite J, it is possible, only in the case $u \to \pm\infty$, *to find expressions involving only* α *or* ξ *and not* ϕ *or* χ. *The case* $J \to \infty$, i.e. the boson case, is exceptional since, as explained, there are two different ways of handling the $\hat{J}_z$ operator, giving two different Lie algebra structures and, therefore, two different representations. But, since $\hat{N} = a^\dagger a$ is obtained from the other generators a and $a^\dagger$, the variable associated with it is redundant.

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